On some estimators of the Hurst index of the solution of SDE driven by a fractional Brownian motion

K. Kubilius[1,*], V. Skorniakov[1,2,**], and D. Melichov[3,†]

[1]Vilnius University, Institute of Mathematics and Informatics, Akademijos 4, LT-08663, Vilnius, Lithuania

[2]Vilnius University, Faculty of Mathematics and Informatics, Naugarduko 24, LT-03225, Vilnius, Lithuania

[3]Vilnius Gediminas Technical University, Saulėtekio al. 11, LT-10223, Vilnius, Lithuania

Abstract

Strongly consistent and asymptotic normal estimators of the Hurst index of a stochastic differential equation driven by a fractional Brownian motion are proposed. The estimators are based on discrete observations of the underlying process.

Keywords: fractional Brownian motion, stochastic differential equation, Hurst index, consistent estimator

1 Introduction

Recently, much attention has been paid to the study of almost sure convergence and asymptotic normality of the first and second order quadratic variations of a wide class of processes with Gaussian increments (see [?], [?], [?], [?], [?], [?], [?], [?]). The conditions for this type of convergence are expressed in terms of covariances of a Gaussian process and depend on some parameter $\gamma \in (0, 2)$. These results help us to obtain the estimate of the parameter γ and to study its asymptotic properties. If the process under consideration is the fractional Brownian motion (fBm) then this parameter is the Hurst index.

The aim of this paper is to obtain a strongly consistent and asymptotically normal Hurst index estimator when the solution of a stochastic differential equation (SDE) driven by a fBm with the Hurst parameter $1/2 < H < 1$ is observed at discrete points. For the first time, to our knowledge, the estimator of the Hurst parameter of a pathwise solution of a SDE driven by a fBm and its asymptotic in some special cases were considered in [?]. A more general situation was considered in [?]. Estimates for the Hurst parameter were constructed according to first- and second-order quadratic variations of the observed values of the solution. Only a strong consistency of these estimates was proved. The rate of convergence of these estimates to the true value of a parameter was obtained in [?], when the maximum of lengths of the interval partition tends to zero.

In this paper we offer estimators which are different from the estimators used in previous papers. In general, these new estimators allow us to obtain not only strongly consistent but also asymptotically normal estimators.

The paper is organized in the following way. In section 2, we present the main results of the paper. Section 3 is devoted to several known results needed for the proofs. Section 5 contains the proofs. Finally, in section 6 some simulations are given in order to illustrate the obtained result.

2 Main result

Consider a stochastic differential equation

$$X_t = \xi + \int_0^t f(X_s)\, ds + \int_0^t g(X_s)\, dB_s^H, \quad t \in [0, T],\ T > 0, \tag{2.1}$$

*Corresponding author. E-mail: kestutis.kubilius@mii.vu.lt

**E-mail: viktor.skorniakov@mif.vu.lt

†E-mail: dmelichov@gmail.com

where f and g are measurable functions, $B^H = \{B_t^H\colon\ 0 \leqslant t \leqslant T\}$ is a fractional Brownian motion (fBm) with Hurst index $1/2 < H < 1$, ξ is a random variable. It is well-known that almost all sample paths of B^H have bounded p-variations for $p > 1/H$.

For $0 < \alpha \leqslant 1$, $\mathcal{C}^{1+\alpha}(\mathbb{R})$ denotes the set of all $\mathcal{C}^1$-functions $g\colon \mathbb{R} \to \mathbb{R}$ such that

$$\sup_x |g'(x)| + \sup_{x\neq y} \frac{|g'(x)-g'(y)|}{|x-y|^\alpha} < \infty\,.$$

Let $(\Omega, \mathcal{F}, \mathbf{P}, \mathbb{F})$, $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ be a given filtered probability space, and let ξ be $\mathcal{F}_0$-measurable r.v. Assume that f be a Lipschitz function and $g \in \mathcal{C}^{1+\alpha}(\mathbb{R})$, $\frac{1}{H} - 1 < \alpha \leqslant 1$. It is well known (see [?], [?], [?], and [?]) that for the fixed fBm $(B^H, \mathbb{F})$, $1/2 < H < 1$, under the above assumptions, the right-hand side of (2.1) is well defined and there exists a unique $\mathbb{F}$ adapted process $X = (X_t)_{t\in[0;T]}$ with almost all sample paths in the class of all continuous functions defined on $[0,T]$ with bounded p-variation for any $p > \frac{1}{H}$ and such that

- $X_0 = \xi$ a.s.,
- $\mathrm{P}\left(\int_0^t |f(X_s)| \d s + \left|\int_0^t g(X_s) \d B_s^H\right| < \infty\right) = 1$ for all $t \in [0;T]$.

We first construct a strongly consistent and asymptotically normal estimate of H if we have additional information about function g in (2.1) and observed values of the solution on discrete points.

Theorem 2.1. *Assume that function g is known and there exist a random variable ς such that $\mathrm{P}(\varsigma < \infty) = 1$ and*

$$\sup_{t\in[0,T]} \frac{1}{|g(X_t)|} \leqslant \varsigma \quad a.s.$$

Then

$$\widehat{H}_n \to H \quad a.s.,$$
$$2\sqrt{n}\,\ln\frac{n}{T}\,(\widehat{H}_n - H) \xrightarrow{d} N(0;\sigma^2) \quad for\ H \in (1/2,1)$$

with a known variance σ^2 defined in Subsection 3.1, where

$$\widehat{H}_n = \varphi_{n,T}^{-1}\left(\frac{1}{n}\sum_{i=2}^n \left(\frac{\Delta^{(2)} X_{\frac{i}{n}T}}{g\big(X_{\frac{i-1}{n}T}\big)}\right)^2\right) \qquad for\ n > T, \qquad \Delta^{(2)} X_{\frac{i}{n}T} = X_{\frac{i}{n}T} - 2X_{\frac{i-1}{n}T} + X_{\frac{i-2}{n}T},$$
$$\varphi_{n,T}(x) = \left(\frac{T}{n}\right)^{2x}(4 - 2^{2x}), \qquad \varphi_{n,T}^{-1}\ is\ the\ inverse\ function\ of\ \varphi_{n,T}, \quad x \in (0,1),\ n > T.$$

It is natural to try to find conditions when it is possible to refuse restrictions of the previous theorem. For this purpose we need some additional definitions. We assume that the process X is observed at time points $\frac{k}{m_n}T$, $k = 1, \ldots, m_n$, where m_n is an increasing sequence of natural numbers defined as follows. Take measurable non-decreasing function $\varphi : (0;\infty) \to (0;\infty)$ which satisfies condition

$$\forall \delta > 0 \quad \varphi(x) = o(x^\delta), \quad x \to \infty.$$

For example, one can take take $\varphi(x) = 1$ or $\varphi(x) = \ln^\alpha n$, $\alpha > 0$. Let's assume, it is known that $H \in (\beta; 1)$ for some $\beta \in (\frac{1}{2}; 1)$. Then we take $k_n = [n^{2\beta}\varphi(n)]$. If we only know that $H \in (1/2, 1)$ then we set $k_n = [n\varphi(n)]$. Put $m_n = nk_n$.

Next denote

$$h(t) = g^2(X_t),\ \kappa = \sup_{t\in[0;T]} h(t);\ I_n = \{-k_n + 2j : j = 2, \ldots, k_n\},$$
$$W_{n,k}^{(1)} = \sum_{j=-k_n+2}^{k_n} \left(X_{t_j^m + \frac{k}{n}T} - 2X_{t_{j-1}^m + \frac{k}{n}T} + X_{t_{j-2}^m + \frac{k}{n}T}\right)^2, \tag{2.2}$$
$$W_{n,k}^{(2)} = \sum_{j\in I_n} \left(X_{t_j^m + \frac{k}{n}T} - 2X_{t_{j-2}^m + \frac{k}{n}T} + X_{t_{j-4}^m + \frac{k}{n}T}\right)^2,$$

where $1 \leqslant k \leqslant n-1$ and $t_j^m = \frac{j}{m_n}T$. For each $n \geqslant 1$ define $A_n = \left\{\frac{1}{n}T, \frac{2}{n}T, \ldots, \frac{n-1}{n}T\right\}$. Finally, set

$$H_n(k) = \frac{1}{2} - \frac{1}{2\ln 2}\ln\frac{W_{n,k}^{(1)}}{W_{n,k}^{(2)}}, \qquad k = 1, \ldots, n-1. \tag{2.3}$$

The following theorem holds.

Theorem 2.2. *Assume that $\mathrm{P}(\kappa > 0) = 1$ and a sequence of random variables (τ_n) has the following properties:*

(i) $\tau_n \in A_n \ \forall n$;

(ii) $\mathrm{P}(\liminf h(\tau_n) > 0) = 1$.

Let $i_n = \frac{n}{T}\tau_n$. *Then*

$$H_n(i_n) \to H \quad a.s. \qquad and \qquad 2\ln 2\sqrt{k_n}\,(H_n(i_n) - H) \xrightarrow{d} N(0;\sigma_H^2) \tag{2.4}$$

with a known variance σ_H^2 *defined in Subsection 3.1.*

Let us give several remarks on the obtained result.

REMARK 2.1. Note that we impose very mild assumption on the model since condition $\kappa > 0$ a.s. simply states that for almost all sample paths function $t \mapsto g(X_t)$ is not trivial one.

REMARK 2.2. Considering theorem 2.2 it is natural to pose question how to construct a sequence satisfying (ii). Proposition 2.1 gives several possible solutions however one might consider the other alternatives. It is straightforward to observe that requirement (ii) is nothing more than a statement that for almost all sample paths of the sequence $(h(\tau_n))$ has to be bounded away from zero. Probably the most simple case is the case when $h(t)$ is a.s. bounded away from zero by a non-random constant. Then one can fix any $t_0 \in (0;T)$ and define (τ_n) by relation

$$|\tau_n - t_0| = \min_{1\leqslant k\leqslant n-1}\left|\frac{kT}{n} - t_0\right|.$$

Then $\tau_n \to t_0$ as $n \to \infty$ and conditions (i)-(ii) are satisfied.

PROPOSITION 2.1. *For each* $n \geqslant 1$ *put* $\overline{W}_n^{(m)} = \frac{1}{n-1}\sum_{j=1}^{n-1} W_{n,j}^{(m)}$, $m = 1, 2$, *and define*

$$\begin{aligned}
i_n^{(1)} &= \arg\max_{1\leqslant k\leqslant n-1} W_{n,k}^{(1)};\\
i_n^{(2)} &= \arg\min_{1\leqslant k\leqslant n-1}\left|\frac{W_{n,k}^{(1)}}{\overline{W}_n^{(1)}} - 1\right|;\\
i_n^{(3)} &= \arg\min_{1\leqslant k\leqslant n-1}\left|\frac{W_{n,k}^{(1)}}{\overline{W}_n^{(1)}} - 1\right| + \left|\frac{W_{n,k}^{(2)}}{\overline{W}_n^{(2)}} - 1\right|;\\
i_n^{(4)} &= \arg\min_{1\leqslant k\leqslant n-1}\left|\frac{\overline{W}_n^{(1)} - W_{n,k}^{(1)}}{\overline{W}_n^{(2)}}\right| + \left|\frac{\overline{W}_n^{(2)} - W_{n,k}^{(2)}}{\overline{W}_n^{(1)}}\right|;\\
\tau_n^{(j)} &= \frac{i_n^{(j)}}{n}\,T.
\end{aligned}$$

If $\mathrm{P}(\kappa > 0) = 1$ *then the sequences of random variables* $(\tau_n^{(j)})$, $j = 1, \ldots, 4$, *satisfy conditions (i) and (ii).*

REMARK 2.3. Note that in usual case the sets $\arg\max$ and $\arg\min$ in definition of $i_n^{(j)}$ consist of one integer point $k \in \{1, 2, \ldots, n\}$. If this is not the case we assume that $i_n^{(j)}$ is equal to the smallest value of the corresponding arg set.

3 Preliminaries

3.1 Several results on fBm

Let $\Delta^{(2)}B^H_{\frac{i}{n}} = B^H_{\frac{i}{n}} - 2B^H_{\frac{i-1}{n}} + B^H_{\frac{i-2}{n}}, i = 2, \ldots, n$ and $\Delta^{(2)}B^H(n) = \left(\Delta^{(2)}B^H_{\frac{2}{n}}, \Delta^{(2)}B^H_{\frac{3}{n}}, \ldots, \Delta^{(2)}B^H_{\frac{n}{n}}\right)$, i.e. $\Delta^{(2)}B^H(n)$ is a centered Gaussian vector of second order increments. We denote by $\lambda_{i,n}, i = 2, \ldots, n$, the eigenvalues of the corresponding covariance matrix whereas λ_n^* stands for a maximal eigenvalue. Applying standard multivariate theory one obtains

$$\mathrm{E}\|\Delta^{(2)}B^H(n)\|^2 = \sum_{i=2}^n \lambda_{i,n} = (n-1)\frac{4-2^{2H}}{n^{2H}}, \qquad \mathrm{Var}\|\Delta^{(2)}B^H(n)\|^2 = 2\sum_{i=2}^n \lambda_{i,n}^2,$$

where $\|x\| = \sqrt{x_1^2 + \cdots + x_k^2}$, $x \in \mathbb{R}^k$, is a standard Euclidian norm. In [?] it was shown (see also [?]) that

$$\mathrm{Var}\|\Delta^{(2)}B^H(n)\|^2 = O\left(\frac{1}{n^{4H-1}}\right) \quad \text{and} \quad \lambda_n^* = O\left(\frac{1}{n^{2H}}\right), \quad n \to \infty. \tag{3.1}$$

In what follows we make use of quantity

$$\widetilde{V}_{n,T} = \frac{n^{2H-1}}{4-2^{2H}}\sum_{i=2}^n \left(T^{-H}\Delta^{(2)}B^H_{\frac{i}{n}T}\right)^2. \tag{3.2}$$

By self-similarity of fBm one deduces that

$$\widetilde{V}_{n,T} \stackrel{d}{=} \frac{n^{2H-1}}{4-2^{2H}} \left\|\Delta^{(2)} B^H(n)\right\|^2.$$

The latter relationship enables us to estimate deviation of $\widetilde{V}_{n,T}$ from 1. To do this we apply inequality[1] given below. It follows from (3.1) and main result of [?]

$$\begin{aligned}
&\mathrm{P}\left(\left|\|\Delta^{(2)} B^H(n)\|^2 - \mathrm{E}\|\Delta^{(2)} B^H(n)\|^2\right| > \varepsilon\right) \\
&\leqslant 2\exp\left[-\min\left(\frac{C_1\varepsilon}{\lambda_n^*}, \frac{C_2\varepsilon^2}{\sum_{i=2}^n \lambda_{i,n}^2}\right)\right] \leqslant 2\exp\left[-\min\left(\widetilde{C}_1\varepsilon n^{2H}, \widetilde{C}_2\varepsilon^2 n^{4H-1}\right)\right]
\end{aligned}$$

for large n. In view of the above we have the bound

$$\begin{aligned}
\mathrm{P}\left(\left|\widetilde{V}_{n,T} - \frac{n-1}{n}\right| > \varepsilon\right) =& \mathrm{P}\left(\left|\|\Delta^{(2)} B^H(n)\|^2 - \mathrm{E}\|\Delta^{(2)} B^H(n)\|^2\right| > \frac{(4-2^{2H})\varepsilon}{n^{2H-1}}\right) \\
\leqslant& 2\exp\left[-\min\left(\widetilde{C}_1\varepsilon(4-2^{2H})n, \widetilde{C}_2\varepsilon^2(4-2^{2H})^2 n\right)\right] \leqslant 2\exp\left[-\widetilde{C}_2\varepsilon^2(4-2^{2H})n\right], \qquad (3.3)
\end{aligned}$$

valid for $\varepsilon \in \left(0; \frac{\widetilde{C}_1}{\widetilde{C}_2} \wedge 1\right)$. The other important statements was also proved in [?] (see also [?]). It asserts that

$$\sqrt{n}\begin{pmatrix} \widetilde{V}_{n,T} - 1 \\ \widetilde{V}_{2n,T} - 1 \end{pmatrix} \xrightarrow{d} N(0; \Sigma_H), \tag{3.4}$$

with a positive definite matrix

$$\Sigma_H = \begin{pmatrix} \sigma^2 & \sigma_*^2 \\ \sigma_*^2 & \sigma^2/2 \end{pmatrix}. \tag{3.5}$$

The explicit expression for Σ_H can be obtained from [?]. It was also given in [?]. To avoid the reader from doing tedious calculations and make paper more self-contained we restate the expression given in [?]:

$$\begin{aligned}
c_1(H) &= \left(\frac{2H(2H-1)(2H-2)(2H-3)}{4-2^{2H}}\right)^2; \qquad c_2(H) = \left(\frac{2^{2H+2}-7-3^{2H}}{4-2^{2H}}\right)^2; \\
\rho_\gamma(l) &= \frac{|l-2|^{2-\gamma} - 4|l-1|^{2-\gamma} + 6|l|^{2-\gamma} - 4|l+1|^{2-\gamma} + |l+2|^{2-\gamma}}{(\gamma-2)(\gamma-1)(\gamma+1)}, \quad l \in \mathbb{Z},\ \gamma \in (0,1); \\
\sigma_1^2 &= \frac{c_2(H)}{2} + c_1(H)\sum_{l=2}^{\infty} \rho_{2-2H}(l)\,\rho_{2-2H}(l-2); \qquad \sigma_2^2 = 2\sqrt{c_2(H)} + c_1(H)\sum_{l=2}^{\infty} \rho_{2-2H}(l)\,\rho_{2-2H}(l-1); \\
\sigma^2 &= 2 + c_2(H) + c_1(H)\sum_{l=2}^{\infty}\left(\rho_{2-2H}(l)\right)^2; \qquad \sigma_*^2 = 2^{-2H}\left(3\sigma^2 + \sigma_1^2 + 4\sigma_2^2\right).
\end{aligned}$$

Remark 3.1. Note that application of the delta method to (3.4) implies

$$\sqrt{n}\left(\frac{\widetilde{V}_{2n,T}}{\widetilde{V}_{n,T}} - 1\right) \xrightarrow{d} N(0; \sigma_H^2) \quad \text{and} \quad \sqrt{n}\left(\ln\frac{\widetilde{V}_{2n,T}}{\widetilde{V}_{n,T}} - \ln 1\right) \xrightarrow{d} N(0; \sigma_H^2) \quad \text{with } \sigma_H^2 = \frac{3}{2}\sigma^2 - 2\sigma_*^2. \tag{3.6}$$

3.2 Variation

In this subsection we remind several facts about p–variation. For details we refer the reader to [?]. Recall that p–variation on $[a;b]$ of a measurable function $f : [a;b] \to \mathbb{R}$ is a quantity

$$v_p(f; [a;b]) = \sup_{\varkappa} \sum_{k=1}^n |f(x_k) - f(x_{k-1})|^p,$$

where $p \in (0;\infty)$ is a fixed number and supremum is taken over all possible partitions

$$\varkappa : a = x_0 < x_1 < \cdots < x_n = b, \qquad n \geqslant 1.$$

By $BV_p([a;b])$ denote the set of all measurable functions $f : [a;b] \to \mathbb{R}$ having finite p–variation on $[a;b]$. In the sequel we use one well known relationship, namely,

$$f, h \in BV_p([a;b]) \implies \left|\int_a^b f\dot{h} - f(y)[h(b)-h(a)]\right| \leqslant C_{p,q} V_p(f;[a;b]) V_q(h;[a;b]) \tag{3.7}$$

[1] inequality is valid for all $\varepsilon > 0$; constants C_1, C_2 are positive and do not depend on n

with $C_{p,q} = \zeta(p^{-1}+q^{-1}) = \sum_{n\geqslant 1} n^{-(p^{-1}+q^{-1})}$, $\frac{1}{p}+\frac{1}{q}>1$, and $V_\gamma(g;[a;b]) = (v_r(g;[a;b]))^{\frac{1}{r}}$.

We also make use of the fact that for each $\varepsilon \in (0;1/2)$ almost all sample paths of fBm $B^H = (B_t^H, t\in[0;T])$ belong to $BV_{H_\varepsilon}([0;T])$, where $H\in(1/2;1)$, $H_\varepsilon = \frac{1}{H-\varepsilon}$, and there exists almost surely finite r.v. $G_{\varepsilon,T}$ with the following property:

$$V_{H_\varepsilon}(B^H;[s;t]) \leqslant G_{\varepsilon,T}|t-s|^{H-\varepsilon}, \qquad \forall s,t\in[0;T] \text{ a.s.}$$

Note that the last inequality implies the same relationship for the initial process X, i.e. there exists almost surely finite r.v. $L_{\varepsilon,T}$ for which

$$V_{H_\varepsilon}(X;[s;t]) \leqslant L_{\varepsilon,T}|t-s|^{H-\varepsilon}, \qquad \forall s,t\in[0;T] \text{ a.s.}$$

4 Proof of Theorem 2.1

Before presenting the proof of this theorem, we want to make few comments on the notion of symbols O and o. Let (Y_n) be a sequence of r.v. and an increasing sequence of real numbers $(h(n))$. Assume that there exists a.s. finite r.v. ς with the property

$$|Y_n| \leqslant \frac{\varsigma}{h(n)}.$$

In this case we write $Y_n = O(h^{-1}(n))$. If one has an array $(Y_{n,k}),\ k=1,\dots,n,\ n\geqslant 1$, then $Y_{n,k} = O(h^{-1}(n))$ means that there exists a.s. finite r.v. ς satisfying

$$\max_{1\leqslant k\leqslant n}|Y_{n,k}| \leqslant \frac{\varsigma}{h(n)}\,.$$

If $Y_n = \varsigma_n\cdot h^{-1}(n)$ and $\varsigma_n\to 0$ a.s. as $n\to\infty$, we write $Y_n = o(h^{-1}(n))$. Similarly, if $\max_{1\leqslant k\leqslant n}|Y_{n,k}| = \varsigma_n\cdot h^{-1}(n)$ and $\varsigma_n\to 0$ a.s. as $n\to\infty$, we write $Y_{n,k} = o(h^{-1}(n))$.

Lemma 4.1. *Set $t_i = \frac{i}{n}T$. The following (expansion, decomposition, asymptotic) relation*

$$\Delta^{(2)}X_{t_i} = X_{t_i} - 2X_{t_{i-1}} + X_{t_{i-2}} = g(X_{t_{i-1}})\Delta^{(2)}B_{t_i}^H + O\left(\frac{1}{n^{2(H-\varepsilon)}}\right), \qquad i=2,\dots n. \tag{4.1}$$

holds

Proof. Since

$$\Delta^{(2)}X_{t_i} = \int_{t_{i-1}}^{t_i} f(X_s)ds - \int_{t_{i-2}}^{t_{i-1}} f(X_s)ds + \int_{t_{i-1}}^{t_i} g(X_s)dB_s^H - \int_{t_{i-2}}^{t_{i-1}} g(X_s)dB_s^H, \qquad i=2,\dots,n,$$

an application of Lipschitz-continuity of f and boundedness of variation of X yields

$$\begin{aligned}\left|\int_{t_{i-1}}^{t_i} f(X_s)ds - \int_{i-2}^{t_{i-1}} f(X_s)ds\right| &\leqslant \int_{t_{i-1}}^{t_i} \left|f(X_s)-f(X_{t_{i-1}})\right|ds - \int_{t_{i-2}}^{t_{i-1}}\left|f(X_{t_{i-1}})-f(X_s)\right|ds\\ &\leqslant \frac{2TK}{n}\max_{1\leqslant i\leqslant n}\sup_{t_{i-1}\leqslant s\leqslant t_i}|X_s - X_{t_{i-1}}| \leqslant KL_{\varepsilon,T}\left(\frac{2T}{n}\right)^{1+(H-\varepsilon)},\end{aligned}$$

where K is a Lipschitz constant and $L_{\varepsilon,T}$ is a r.v. bounding variation of X (see Section 3).

Next we observe that

$$\begin{aligned}\int_{t_{i-1}}^{t_i} g(X_s)dB_s^H - \int_{t_{i-2}}^{t_{i-1}} g(X_s)dB_s^H = &\int_{t_{i-1}}^{t_i} g(X_s)-g(X_{t_{i-1}})dB_s^H - \int_{t_{i-2}}^{t_{i-1}} g(X_s)-g(X_{t_{i-1}})dB_s^H\\ &+ g(X_{t_{i-1}})\left(\Delta B_{t_i}^H - \Delta B_{t_{i-1}}^H\right), \qquad i=2,\dots,n.\end{aligned}$$

By (3.7)

$$\begin{aligned}\left|\int_{t_{i-1}}^{t_i} g(X_s)-g(X_{t_{i-1}})dB_s^H\right| &\leqslant \|g'\|_\infty C_{H_\varepsilon,H_\varepsilon}V_{H_\varepsilon}(X;[t_{i-1};t_i])\,V_{H_\varepsilon}\left(B^H;[t_{i-1};t_i]\right)\\ &\leqslant \|g'\|_\infty C_{H_\varepsilon,H_\varepsilon}L_{\varepsilon,T}G_{\varepsilon,T}\left(\frac{T}{n}\right)^{2(H-\varepsilon)},\end{aligned}$$

where $L_{\varepsilon,T}, G_{\varepsilon,T}$ are a.s. finite r.v. defined in Subsection 3.2 and $C_{H_\varepsilon,H_\varepsilon}$ is also finite provided ε is sufficiently small. The same inequality holds for the term $\int_{t_{i-2}}^{t_{i-1}}[g(X_s)-g(X_{t_{i-1}})]dB_s^H$. Hence (4.1) is true.

Proof of Theorem 2.1. Observe first that the function

$$\varphi_{n,T}(x) = \Big(\frac{T}{n}\Big)^{2x}(4 - 2^{2x}), \qquad x \in (0,1),$$

is continuous and strictly decreasing for $n > T$. Thus it has the inverse function $\varphi_{n,T}^{-1}$ for $n > T$. By Lemma 4.1 we obtain

$$\begin{aligned}
\frac{\varphi_{n,T}(\widehat{H}_n)}{\varphi_{n,T}(H)} =& \Big[\Big(\frac{T}{n}\Big)^{2H}(4-2^{2H})\Big]^{-1}\varphi_{n,T}\left(\varphi_{n,T}^{-1}\left(\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\Delta^{(2)}X_{\frac{i}{n}T}}{g\big(X_{\frac{i-1}{n}T}\big)}\right)^2\right)\right)\\
=& \Big[\Big(\frac{T}{n}\Big)^{2H}(4-2^{2H})\Big]^{-1}\left(\frac{1}{n}\sum_{i=2}^{n}\left(\frac{\Delta^{(2)}X_{\frac{i}{n}T}}{g\big(X_{\frac{i-1}{n}T}\big)}\right)^2\right)\\
=& \frac{n^{2H-1}}{\big(\frac{T}{n}\big)^{2H}(4-2^{2H})}\left(\sum_{i=1}^{n}\Big(\Delta^{(2)}B^H_{\frac{i}{n}T}\Big)^2 + O\Big(\frac{1}{n^{3(H-\varepsilon)-1}}\Big)\right)\\
=& \frac{n^{2H-1}}{4-2^{2H}}\sum_{i=1}^{n}\Big(T^{-H}\Delta^{(2)}B^H_{\frac{i}{n}T}\Big)^2 + O\Big(\frac{1}{n^{H-3\varepsilon}}\Big).
\end{aligned} \tag{4.2}$$

Note that

$$\frac{\varphi_{n,T}(\widehat{H}_n)}{\varphi_{n,T}(H)} \longrightarrow 1 \qquad \text{a.s.} \quad \text{as } n \to \infty.$$

Thus for every $\delta > 0$ there exist n_0 such that

$$\varphi_{n,T}(H+\delta) < \varphi_{n,T}(\widehat{H}_n) < \varphi_{n,T}(H-\delta).$$

for $n \geqslant n_0$, i.e.

$$\varphi_{n,T}(\widehat{H}_n) \in \big(\varphi_{n,T}(H+\delta), \varphi_{n,T}(H-\delta)\big) \qquad \forall n \geqslant n_0.$$

The function $\varphi_{n,T}$ is strictly decreasing for all $n > T$. Thus $\widehat{H}_n \in (H-\delta, H+\delta)$ for all $n \geqslant n_0$ and it is strongly consistence.

Now we prove asymptotic normality of $\widehat{H}_n$. Note that

$$\begin{aligned}
\ln\left(\frac{\varphi_{n,T}(\widehat{H}_n)}{\varphi_{n,T}(H)}\right) =& \ln\Big(\frac{n}{T}\Big)^{2(\widehat{H}_n-H)} + \ln\frac{4-2^{2\widehat{H}_n}}{4-2^{2H}}\\
=& -2(\widehat{H}_n - H)\ln\Big(\frac{n}{T}\Big) + \ln\big(4-2^{2\widehat{H}_n}\big) - \ln\big(4-2^{2H}\big).
\end{aligned} \tag{4.3}$$

By Lagrange theorem for the function $h(x) = \ln(4-2^{2x})$, $x \in (0,1)$, we obtain

$$\ln\big(4-2^{2\widehat{H}_n}\big) - \ln\big(4-2^{2H}\big) = (\widehat{H}_n - H)\int_0^1 h'\big(H + t(\widehat{H}_n - H)\big)dt$$

The equality (5.1) we rewrite in a following way

$$2\sqrt{n}\,\ln\frac{n}{T}\,(\widehat{H}_n - H) = \frac{-\sqrt{n}\Big(\ln\Big(\frac{\varphi_{n,T}(\widehat{H}_n)}{\varphi_{n,T}(H)}\Big) - \ln 1\Big)}{1+o(1)}.$$

It follows from (3.4) and Remark 3.1 that

$$\begin{aligned}
\sqrt{n}\Big(\frac{\varphi_{n,T}(\widehat{H}_n)}{\varphi_{n,T}(H)} - 1\Big) =& \sqrt{n}\left(\frac{n^{2H-1}}{4-2^{2H}}\sum_{i=2}^{n}\Big(T^{-H}\Delta^{(2)}B^H_{\frac{i}{n}T}\Big)^2 - 1\right) + O\Big(\frac{1}{n^{H-3\varepsilon-1/2}}\Big)\\
&\xrightarrow{d} N(0;\sigma^2)
\end{aligned}$$

for sufficiently small ε.

5 Proof of Theorem 2.2

In what follows we use notions already introduced in Section 2 without restatement. In order to prove our main results we need two auxiliary lemmas. Let

$$V^{(1)}_{n,T}(k) = \frac{(2k_n)^{2H-1}}{T^{2H}(4-2^{2H})}\sum_{j=-k_n+2}^{k_n}\Big(B^H_{t^m_j+\frac{k}{n}T} - 2B^H_{t^m_{j-1}+\frac{k}{n}T} + B^H_{t^m_{j-2}+\frac{k}{n}T}\Big)^2;$$

$$V^{(2)}_{n,T}(k) = \frac{k_n^{2H-1}}{T^{2H}(4-2^{2H})}\sum_{j\in I_n}\Big(B^H_{t^m_j+\frac{k}{n}T} - 2B^H_{t^m_{j-2}+\frac{k}{n}T} + B^H_{t^m_{j-4}+\frac{k}{n}T}\Big)^2;$$

with $1 \leqslant k \leqslant n-1$.

Lemma 5.1. *The following relations hold*

$$\max_{1\leqslant k\leqslant n-1}\left|\frac{V^{(1)}_{n,T}(k)}{V^{(2)}_{n,T}(k)}-1\right|\longrightarrow 0 \qquad a.s.;$$

$$\sqrt{k_n}\left(\frac{V^{(1)}_{n,T}(k)}{V^{(2)}_{n,T}(k)}-1\right)\xrightarrow{d} N(0;\sigma_H^2), \qquad \textit{for each } k=1,\dots,n-1;$$

$$\max_{1\leqslant k\leqslant n-1}\left|V^{(p)}_{n,T}(k)-1\right|\longrightarrow 0 \qquad a.s.\ \textit{for } p=1,2;$$

$$\max_{k,l\in\{1,\dots n-1\}}\left|\frac{V^{(p)}_{n,T}(k)}{V^{(p)}_{n,T}(l)}-1\right|\longrightarrow 0 \qquad a.s.\ \textit{for } p=1,2;$$

where σ_H^2 is the same as in (3.6).

Proof. We will give the proofs of the first two assertions. The proof of the rest is obtained by obvious alterations.

By self-similarity and stationarity of first order increments

$$\frac{V^{(1)}_{n,T}(k)}{2^{2H-1}V^{(2)}_{n,T}(k)}\stackrel{d}{=}\frac{n^{2H}\sum_{j=-k_n+2}^{k_n}\left(B^H_{\frac{j+k_n}{m_n}}-2B^H_{\frac{j+k_n-1}{m_n}}+B^H_{\frac{j+k_n-2}{m_n}}\right)^2}{n^{2H}\sum_{j\in I_n}\left(B^H_{\frac{j+k_n}{m_n}}-2B^H_{\frac{j+k_n-2}{m_n}}+B^H_{\frac{j+k_n-4}{m_n}}\right)^2}$$

$$\stackrel{d}{=}\frac{\sum_{j=-k_n+2}^{k_n}\left(B^H_{\frac{j}{k_n}+1}-2B^H_{\frac{j-1}{k_n}+1}+B^H_{\frac{j-2}{k_n}+1}\right)^2}{\sum_{j\in I_n}\left(B^H_{\frac{j}{k_n}+1}-2B^H_{\frac{j-2}{k_n}+1}+B^H_{\frac{j-4}{k_n}+1}\right)^2}\stackrel{d}{=}\frac{2^{-2H}\sum_{j=2}^{2k_n}\left(B^H_{\frac{j}{k_n}}-2B^H_{\frac{j-1}{k_n}}+B^H_{\frac{j-2}{k_n}}\right)^2}{2^{-2H}\sum_{j=2}^{k_n}\left(B^H_{\frac{2j}{k_n}}-2B^H_{\frac{2(j-1)}{k_n}}+B^H_{\frac{2(j-2)}{k_n}}\right)^2}$$

$$\stackrel{d}{=}\frac{\sum_{j=2}^{2k_n}\left(B^H_{\frac{j}{2k_n}}-2B^H_{\frac{j-1}{2k_n}}+B^H_{\frac{j-2}{2k_n}}\right)^2}{\sum_{j=2}^{k_n}\left(B^H_{\frac{j}{k_n}}-2B^H_{\frac{j-2}{k_n}}+B^H_{\frac{j-4}{k_n}}\right)^2}=\frac{\widetilde{V}_{2k_n,1}}{2^{2H-1}\widetilde{V}_{k_n,1}},$$

with $\widetilde{V}_{2k_n,1},\widetilde{V}_{k_n,1}$ defined by (3.2). Hence

$$\frac{V^{(1)}_{n,T}(k)}{V^{(2)}_{n,T}(k)}\stackrel{d}{=}\frac{\widetilde{V}_{2k_n,1}}{\widetilde{V}_{k_n,1}} \qquad \text{for each } k=1,\dots,n-1.$$

Let $\zeta_{n,T}=\max_{1\leqslant k\leqslant n-1}\left|\frac{V^{(1)}_{n,T}(k)}{V^{(2)}_{n,T}(k)}-1\right|$. Then for all $\varepsilon>0$

$$\mathrm{P}\left(\zeta_{n,T}>\varepsilon\right)\leqslant n\,\mathrm{P}\left(\left|\frac{\widetilde{V}_{2k_n,1}}{\widetilde{V}_{k_n,1}}-1\right|>\varepsilon\right).$$

Let (ε_n) be a sequence of positive numbers such that $\varepsilon_n\downarrow 0$, $n\to\infty$, and let $2k_n^{-1}<\varepsilon_n$. Since $|x/y-1|\leqslant 2\varepsilon_n$ for all $x,y\in(1-2\varepsilon_n;1+2\varepsilon_n)$ application of (3.3) gives

$$\begin{aligned}\mathrm{P}\left(\left|\frac{\widetilde{V}_{2k_n,1}}{\widetilde{V}_{k_n,1}}-1\right|>2\varepsilon_n\right)&\leqslant\mathrm{P}\left(|\widetilde{V}_{2k_n,1}-1|>2\varepsilon_n\right)+\mathrm{P}\left(|\widetilde{V}_{k_n,1}-1|>2\varepsilon_n\right)\\&\leqslant\mathrm{P}\left(\left|\widetilde{V}_{2k_n,1}-\frac{2k_n-1}{2k_n}\right|>\varepsilon_n\right)+\mathrm{P}\left(\left|\widetilde{V}_{k_n,1}-\frac{k_n-1}{k_n}\right|>\varepsilon_n\right)\\&\leqslant 4\exp\left(-C\varepsilon_n^2k_n\right).\end{aligned}$$

Set $\varepsilon_n=(\sqrt{Ck_n})^{-1}\sqrt{3\ln n}$. Then $2k_n^{-1}<\varepsilon_n$ for large n and the series $\sum\mathrm{P}(\zeta_{n,T}>\varepsilon_n)$ converges. Consequently, $\zeta_{n,T}\to 0$ a.s.

We have already proved that the distribution functions of

$$\sqrt{k_n}\left(\frac{V^{(1)}_{n,T}(k)}{V^{(2)}_{n,T}(k)}-1\right)\quad\text{and}\quad\sqrt{k_n}\left(\frac{\widetilde{V}_{2k_n,1}}{\widetilde{V}_{k_n,1}}-1\right),$$

are equal. By Remark 3.1 we obtain the required assertion.

Lemma 5.2. *If $\varepsilon \in (0; 0.3)$ is sufficiently small then the following relations hold:*

$$W_{n,k}^{(1)} = h\left(\frac{k}{n}T\right)\frac{T^{2H}(4-2^{2H})}{(2k_n)^{2H-1}}V_{n,T}^{(1)}(k) + O\left(\frac{1}{n^{H-\varepsilon}k_n^{2H-1}}\right) = h\left(\frac{k}{n}T\right)\frac{T^{2H}(4-2^{2H})}{(2k_n)^{2H-1}} + o\left(\frac{1}{k_n^{2H-1}}\right); \tag{5.1}$$

$$W_{n,k}^{(2)} = h\left(\frac{k}{n}T\right)\frac{T^{2H}(4-2^{2H})}{(k_n)^{2H-1}}V_{n,T}^{(2)}(k) + O\left(\frac{1}{n^{H-\varepsilon}k_n^{2H-1}}\right) = h\left(\frac{k}{n}T\right)\frac{T^{2H}(4-2^{2H})}{(k_n)^{2H-1}} + o\left(\frac{1}{k_n^{2H-1}}\right); \tag{5.2}$$

$$\overline{W}_n^{(1)} = \frac{T^{2H-1}(4-2^{2H})}{(2k_n)^{2H-1}}\int_0^T h(t)\,\underset{.}{t} + o\left(\frac{1}{k_n^{2H-1}}\right); \tag{5.3}$$

$$\overline{W}_n^{(2)} = \frac{T^{2H-1}(4-2^{2H})}{(k_n)^{2H-1}}\int_0^T h(t)\,\underset{.}{t} + o\left(\frac{1}{k_n^{2H-1}}\right); \tag{5.4}$$

with $1 \leqslant k \leqslant n-1$.

Proof. Step 1. In the same way as in Lemma 4.1 one obtains relation

$$\Delta^{(2)}X_{t_i^m} = X_{t_i^m} - 2X_{t_{i-1}^m} + X_{t_{i-2}^m} = g(X_{t_{i-1}^m})\Delta^{(2)}B_{t_i^m}^H + O\left(\frac{1}{m_n^{2(H-\varepsilon)}}\right), \qquad i = 2, \dots m_n. \tag{5.5}$$

Step 2. Note that

$$\begin{aligned}|h(t) - h(s)| =& |g(X_t) - g(X_s)||g(X_t) + g(X_s)| \\ \leqslant& 2\big[|g'|_\infty V_{H_\varepsilon}(X;[0,T]) + |g(\xi)|\big]|g'|_\infty |X_t - X_s| \\ \leqslant& 2\big[|g'|_\infty V_{H_\varepsilon}(X;[0,T]) + |g(\xi)|\big]|g'|_\infty L_{\varepsilon,T}|t-s|^{H-\varepsilon},\end{aligned} \tag{5.6}$$

where $|g'|_\infty = \sup_{t\in[0;T]}|g'(X_t)|$. It follows from relation (5.5), inequality (5.6), and Lemma 5.1 that

$$\begin{aligned}W_{n,k}^{(1)} =& \sum_{j=-k_n+2}^{k_n}\left(\Delta^{(2)}X_{t_j^m+\frac{k}{n}T}\right)^2 = \sum_{j=-k_n+2}^{k_n} h\left(t_{j-1}^m + \frac{k}{n}T\right)\left(\Delta^{(2)}B_{t_j^m+\frac{k}{n}T}^H\right)^2 + O\left(\frac{k_n}{m_n^{3(H-\varepsilon)}}\right) \\ =& h\left(\frac{k}{n}T\right)\sum_{j=-k_n+2}^{k_n}\left(\Delta^{(2)}B_{t_j^m+\frac{k}{n}T}^H\right)^2 + \sum_{j=-k_n+2}^{k_n}\left(h\left(t_{j-1}^m+\frac{k}{n}T\right) - h\left(\frac{k}{n}T\right)\right)\left(\Delta^{(2)}B_{t_j^m+\frac{k}{n}T}^H\right)^2 \\ &+ O\left(\frac{k_n}{m_n^{3(H-\varepsilon)}}\right) \\ =& h\left(\frac{k}{n}T\right)\frac{T^{2H}(4-2^{2H})}{(2k_n)^{2H-1}}\left(\frac{(2k_n)^{2H-1}}{(4-2^{2H})}\sum_{j=-k_n+2}^{k_n}\left(T^{-H}\Delta^{(2)}B_{t_j^m+\frac{k}{n}T}^H\right)^2\right) \\ &+ O\left(\frac{k_n}{n^{H-\varepsilon}m_n^{4(H-\varepsilon)}}\right) + O\left(\frac{k_n}{m_n^{3(H-\varepsilon)}}\right) \\ =& h\left(\frac{k}{n}T\right)\frac{T^{2H}(4-2^{2H})}{(2k_n)^{2H-1}}V_{n,T}^{(1)}(k) + O\left(\frac{1}{n^{H-\varepsilon}k_n^{2H-1}}\right).\end{aligned}$$

Reasoning as in the previous case, we can prove that

$$W_{n,k}^{(2)} = h\left(\frac{k}{n}T\right)\frac{T^{2H}(4-2^{2H})}{(k_n)^{2H-1}}V_{n,T}^{(2)}(k) + O\left(\frac{1}{n^{H-\varepsilon}k_n^{2H-1}}\right).$$

Next we consider the quantity $\overline{W}_n^{(1)}$. Note that

$$\begin{aligned}\overline{W}_n^{(1)} =& \frac{1}{n-1}\sum_{k=1}^{n-1}W_{n,k}^{(1)} = \frac{1}{n-1}\sum_{k=1}^{n-1}h\left(\frac{k}{n}T\right)\frac{T^{2H}(4-2^{2H})}{(2k_n)^{2H-1}}V_{n,T}^{(1)}(k) + O\left(\frac{1}{n^{H-\varepsilon}k_n^{2H-1}}\right) \\ =& \frac{T^{2H}(4-2^{2H})}{(2k_n)^{2H-1}}\left(\frac{1}{n-1}\sum_{k=1}^{n-1}h\left(\frac{k}{n}T\right)\right) + \frac{T^{2H}(4-2^{2H})}{(2k_n)^{2H-1}}\left(\frac{1}{n-1}\sum_{k=1}^{n-1}h\left(\frac{k}{n}T\right)\left[V_{n,T}^{(1)}(k)-1\right]\right) \\ &+ O\left(\frac{1}{n^{H-\varepsilon}k_n^{2H-1}}\right).\end{aligned}$$

Since almost all sample paths of X are continuous, it follows that

$$\frac{T}{n-1}\sum_{k=1}^{n-1}h\left(\frac{k}{n}T\right)\xrightarrow[n\to\infty]{}\int_0^T h(t)\,\underset{.}{t} \qquad \text{a.s.} \tag{5.7}$$

Since

$$\left| \frac{T}{n-1} \sum_{k=1}^{n-1} h\left(\frac{k}{n}T\right)\left[V_{n,T}^{(1)}(k)-1\right] \right| \leqslant \max_{1\leqslant k\leqslant n-1} \left|V_{n,T}^{(1)}(k)-1\right| \cdot \frac{T}{n-1} \sum_{k=1}^{n-1} h\left(\frac{k}{n}T\right) \tag{5.8}$$

then by Lemma 5.1 and (5.7) we get that the right side of (5.8) converges to 0. Thus

$$\overline{W}_n^{(1)} = \frac{T^{2H-1}(4-2^{2H})}{(2k_n)^{2H-1}} \left(\frac{T}{n-1}\sum_{k=1}^{n-1} h\left(\frac{k}{n}T\right)\right) + o\left(\frac{1}{k_n^{2H-1}}\right).$$

and relation (5.3) holds. (5.4) follows similarly. The proof of the lemma is thus complete.

Proof of Theorem 2.2. Fix $\omega \in A = \{\kappa > 0\} \cap \{\liminf h(\tau_n) > 0\}$. Next find $\delta > 0$ and $n_0 < \infty$ for which $h(\tau_n) \geqslant \delta$ whenever $n \geqslant n_0$. It follows from Lemma 5.1 that

$$V_{n,T}^{(p)}(i_n) \xrightarrow[n\to\infty]{} 1, \qquad p = 1, 2.$$

This, together with (5.1) and (5.2), yields

$$\ln W_{n,i_n}^{(1)} = \ln V_{n,T}^{(1)}(i_n) - (2H-1)\ln 2 + \ln\left[h(\tau_n)\frac{T^{2H}(4-2^{2H})}{(k_n)^{2H-1}}\right] + \ln\left(1 + O\left(\frac{1}{n^{H-\varepsilon}}\right)\right)$$

and

$$\ln W_{n,i_n}^{(2)} = \ln V_{n,T}^{(2)}(i_n) + \ln\left[h(\tau_n)\frac{T^{2H}(4-2^{2H})}{(k_n)^{2H-1}}\right] + \ln\left(1 + O\left(\frac{1}{n^{H-\varepsilon}}\right)\right).$$

By applying Maclaurin expansion, we obtain that $\ln\left(1 + O(\frac{1}{n^{H-\varepsilon}})\right) = O(\frac{1}{n^{H-\varepsilon}})$. Thus,

$$H_n(i_n) = \frac{1}{2} - \frac{1}{2\ln 2}\left(\ln W_{n,T}^{(1)}(i_n) - \ln W_{n,T}^{(2)}(i_n)\right) = H - \frac{1}{2\ln 2}\ln\left(\frac{V_{n,T}^{(1)}(i_n)}{V_{n,T}^{(2)}(i_n)}\right) + O\left(\frac{1}{n^{H-\varepsilon}}\right). \tag{5.9}$$

Using Lemma 5.1, we have

$$\frac{V_{n,T}^{(1)}(i_n)}{V_{n,T}^{(2)}(i_n)} \xrightarrow[n\to\infty]{} 1 \qquad \text{a.s.}$$

Since $\mathrm{P}(A) = 1$, the consistency of $H_n(i_n)$ is proved.

Now rewrite equality (5.9) in the following way

$$2\ln 2\sqrt{k_n}\,(H_n(i_n) - H) = -\sqrt{k_n}\ln\left(\frac{V_{n,T}^{(1)}(i_n)}{V_{n,T}^{(2)}(i_n)}\right) + O\left(\frac{\sqrt{\varphi(n)}}{n^{H-\beta-\varepsilon}}\right).$$

For sufficiently small ε the last term goes to 0 a.s. This, together with Lemma 5.1 and the delta method (see Remark 3.1), yields the required statement.

Lemma 5.3. *Set $A = \{\kappa > 0\} \cap \{X$ is continuous$\}$ and let*

$$\tau_0(\omega) = \begin{cases} 0, & \text{for} \quad \omega \in A^c, \\ \inf\{t \in (0,T) : h(t,\omega) = \sup_{s\leqslant T} h(s,\omega)\}, & \text{for} \quad \omega \in A, \end{cases}$$

$$\widehat{\tau}_0(\omega) = \begin{cases} 0, & \text{for} \quad \omega \in A^c, \\ \inf\left\{t \in (0,T) : Th(t,\omega) = \int_0^T h(s,\omega)\,ds\right\}, & \text{for} \quad \omega \in A. \end{cases} \tag{5.10}$$

Then τ_0 and $\widehat{\tau}_0$ are $\mathcal{F}$-measurable r. v's.

Proof First we prove that τ_0 is $\mathcal{F}$-measurable r. v. The proof follows from the equality

$$\{\omega : \tau_0(\omega) \leqslant x\} = \left(\bigcap_{n=1}^{\infty} \bigcup_{q\in\mathbb{Q}\cap[0,x]} \left\{\omega \in A : \left|h(q,\omega) - \sup_{s\leqslant T} h(s,\omega)\right| \leqslant \frac{1}{n}\right\}\right) \cup A^c. \tag{5.11}$$

Now we verify it. Denote the left side of (5.11) by B and the right side by C. First we prove that $B \subset C$. If $\omega \in B \cap A^c$ then it is obvious that $\omega \in C$. If $\omega \in B \cap A$ then $h(\tau_0(\omega),\omega) = \sup_{s\leqslant T} h(s,\omega)$ and there exists a sequence of rational numbers $(q_n) \in \mathbb{Q} \cap [0,x]$ such that $q_n \to \tau_0(\omega)$ and

$$\left|h(q_n,\omega) - h(\tau_0(\omega),\omega)\right| \leqslant \frac{1}{n} \qquad \text{for all } n \geqslant 1.$$

Thus $\omega \in C$ and $B \subset C$.

Now we prove $C \subset B$. Assume that $\omega \in C \cap A$. Then for every $n \geqslant 1$ we can choose $q_n \in \mathbb{Q} \cap [0, x]$ such that

$$\left| h(q_n, \omega) - \sup_{s \leqslant T} h(s, \omega) \right| \leqslant \frac{1}{n}.$$

The sequence (q_n) is bounded. Thus we can choose a subsequence (q_{n_k}) which converges to some limit x_0. It is clear that $x_0 \in [0, x]$. By definition of $\tau_0(\omega)$ we get that $\tau_0(\omega) \leqslant x_0$. Thus $\omega \in B$ and $C \subset B$.

$\mathcal{F}$-measurability of r. v. $\widehat{\tau}_0$ follows from equality

$$\{\omega : \widehat{\tau}_0(\omega) \leqslant x\} = \left(\bigcap_{n=1}^{\infty} \bigcup_{q \in \mathbb{Q} \cap [0,x]} \left\{ \omega \in A : \left| h(q, \omega) - \int_0^T h(s, \omega) ds \right| \leqslant \frac{1}{n} \right\} \right) \cup A^c. \tag{5.12}$$

and Lagrange's mean value theorem.

Proof of proposition 2.1. For each n determine $\tau_n \in A_n = \{\frac{1}{n}T, \dots, \frac{n-1}{n}T\}$ from equality

$$|\tau_n(\omega) - \tau_0(\omega)| = \min_{1 \leqslant k \leqslant n-1} \left| \frac{kT}{n} - \tau_0(\omega) \right|. \tag{5.13}$$

Then $\tau_n \xrightarrow[n \to \infty]{} \tau_0$ and τ_n is $\mathcal{F}$-measurable r. v. for every n. By definition of $i_n^{(1)}$ and (5.1)

$$W^{(1)}_{n, i_n^{(1)}} = h\left(\tau_n^{(1)}\right) \frac{T^{2H}(4 - 2^{2H})}{(2k_n)^{2H-1}} + o\left(\frac{1}{k_n^{2H-1}}\right) \geqslant W^{(1)}_{n, \frac{n}{T}\tau_n} = h\left(\tau_n\right) \frac{T^{2H}(4 - 2^{2H})}{(2k_n)^{2H-1}} + o\left(\frac{1}{k_n^{2H-1}}\right).$$

Multiplying both sides by $\frac{(2k_n)^{2H-1}}{T^{2H}(4-2^{2H})}$ and rearranging terms gives $h\left(\tau_n^{(1)}\right) \geqslant h\left(\tau_n\right) + o(1)$. Consequently,

$$\liminf h\left(\tau_n^{(1)}\right) \geqslant \liminf \left(h\left(\tau_n\right) + o(1)\right) = \kappa > 0.$$

With the remaining sequences $(\tau_n^{(j)})$ we proceed in a similar way. Let $\widehat{\tau}_0$ be a random variable as defined in (5.10). Next we denote $\|h\| = \int_0^T h(t)\, ṭ$. Condition $\kappa > 0$ implies $\|h\| > 0$. A sequence of random variables (τ_n) is defined by using relation (5.13) and replacing τ_0 by $\widehat{\tau}_0$. By using Lemma 5.2, positivity of $\|h\|$, and finiteness of $\sup_{t \leqslant T} h(t)$, we obtain

$$\begin{aligned}
\frac{W^{(p)}_{n,k}}{\overline{W}^{(p)}_n} &= \frac{Th\left(\frac{k}{n}T\right) + o(1)}{\|h\| + o(1)} = \frac{Th\left(\frac{k}{n}T\right)}{\|h\|} + o(1), \qquad p = 1, 2;\\
\frac{W^{(1)}_{n,k}}{\overline{W}^{(2)}_n} &= \frac{\left(\frac{1}{2}\right)^{2H-1} Th\left(\frac{k}{n}T\right) + o(1)}{\|h\| + o(1)} = \left(\frac{1}{2}\right)^{2H-1} \frac{Th\left(\frac{k}{n}T\right)}{\|h\|} + o(1);\\
\frac{W^{(2)}_{n,k}}{\overline{W}^{(1)}_n} &= \frac{Th\left(\frac{k}{n}T\right) + o(1)}{\left(\frac{1}{2}\right)^{2H-1} \|h\| + o(1)} = 2^{2H-1} \frac{Th\left(\frac{k}{n}T\right)}{\|h\|} + o(1);\\
\frac{\overline{W}^{(1)}_n}{\overline{W}^{(2)}_n} &= \frac{1}{2^{2H-1}} + o(1); \qquad \frac{\overline{W}^{(2)}_n}{\overline{W}^{(1)}_n} = 2^{2H-1} + o(1).
\end{aligned}$$

Therefore by definition of $i_n^{(j)}$ and τ_n

$$\begin{aligned}
&\left| \frac{W^{(1)}_{n, i_n^{(2)}}}{\overline{W}^{(1)}_n} - 1 \right| \leqslant \left| \frac{W^{(1)}_{n, \tau_n}}{\overline{W}^{(1)}_n} - 1 \right| = \left| \frac{Th\left(\tau_n\right) - \|h\|}{\|h\|} + o(1) \right|;\\
&\left| \frac{W^{(1)}_{n, i_n^{(3)}}}{\overline{W}^{(1)}_n} - 1 \right| + \left| \frac{W^{(2)}_{n, i_n^{(3)}}}{\overline{W}^{(2)}_n} - 1 \right| \leqslant \left| \frac{W^{(1)}_{n, \tau_n}}{\overline{W}^{(1)}_n} - 1 \right| + \left| \frac{W^{(2)}_{n, \tau_n}}{\overline{W}^{(2)}_n} - 1 \right| = 2 \left| \frac{Th\left(\tau_n\right) - \|h\|}{\|h\|} + o(1) \right|;\\
&\left| \frac{\overline{W}^{(1)}_n - W^{(1)}_{n, i_n^{(4)}}}{\overline{W}^{(2)}_n} \right| + \left| \frac{\overline{W}^{(2)}_n - W^{(2)}_{n, i_n^{(4)}}}{\overline{W}^{(1)}_n} \right| \leqslant \left| \frac{\overline{W}^{(1)}_n - W^{(1)}_{n, \tau_n}}{\overline{W}^{(2)}_n} \right| + \left| \frac{\overline{W}^{(2)}_n - W^{(2)}_{n, \tau_n}}{\overline{W}^{(1)}_n} \right|\\
&\qquad \leqslant \left(2^{2H-1} + \left(\frac{1}{2}\right)^{2H-1} \right) \left| \frac{Th\left(\tau_n\right) - \|h\|}{\|h\|} + o(1) \right|.
\end{aligned}$$

Since h is a continuous process then convergence $\tau_n \to \widehat{\tau}_0$ implies $Th(\tau_n) \to Th(\widehat{\tau}_0) = \|h\|$. Consequently, right-hand side of each inequality vanishes. So does the left-hand side and it remains to observe that in such case $\lim_{n \to \infty} h(\tau_n^{(j)}) = \|h\|$, $j = 2, 3, 4$.

6 Simulations

The goal of this section is to compare the performance of the obtained estimators. The processes chosen for simulations were

I: $X_t = X_0 + \int_0^t \sin(X_t)\, dt + \int_0^t \cos(X_t)\, dB_t^H,\ t \in [0,\ 1],\ X_0 = 1;$

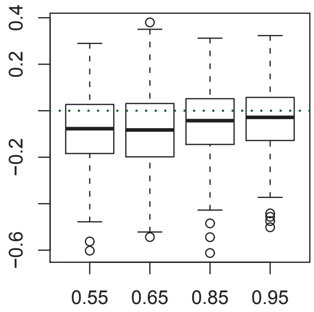
II: $X_t = X_0 + \int_0^t \sin(X_t)\, dt + \int_0^t (2 + \cos(X_t))\, dB_t^H,\ t \in [0,\ 1],\ X_0 = 1.$

The sample paths of the fractional Brownian motion were generated using the circulant matrix embedding (or Wood-Chan) method (see, f. e., [**?**]). The sample paths of the processes defined above were obtained by applying the Milstein time-discrete approximation. The sample path lengths and the values of the Hurst index were chosen to be, respectively, n^2+1 where $n \in \{50,\ 100,\ 150,\ 250,\ 500\}$ and $H \in \{0.55,\ 0.65,\ 0.85,\ 0.95\}$. For each tuple $(n^2+1,\ H)$ we simulated 200 sample paths of both considered processes. The estimators $\widehat{H}_i^n,\ i \in \{1, 2, 3, 4\}$ were calculated using the formula (2.3) with the sequences $i_n^{(\cdot)}$ chosen as described in the Proposition 2.1 and were calculated for the process (**I**). The estimator $\widehat{H}_5^n$ corresponds the statistic of the Theorem 2.1 and, due to the requirement for the function $g(X_t)$ to be separated from zero, it was initially calculated for the process (**II**). However, after comparing the numerical characteristics of $\widehat{H}_5^n$ to those of $\widehat{H}_i^n,\ i \in \{1, 2, 3, 4\}$ we decided to check how it performs in case the requirement for the function $g(X_t)$ is not fulfilled, hence we calculated $\widehat{H}_5^n$ for the process (**I**) as well. For notational simplicity, let $Hi := \widehat{H}_i^n$.

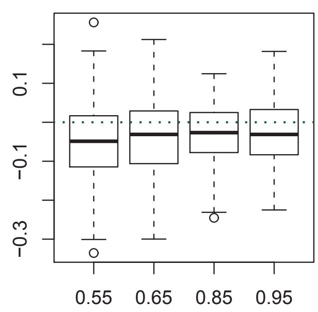
The table 1 presents the mean squared errors and the average absolute deviations of the considered estimators (* – MSE was multiplied by 10^5 instead of 10^3). It can be seen that the numeric characteristics of $H3$ and $H4$ are considerably better than those of $H1$ and $H2$. The estimator $H1$ displayed a rather large and slowly decreasing negative bias. Additionally, $H3$ and $H4$ show similar behavior for $H < 3/4$; in fact, on the average for 80% of the sample paths the estimated values of $H3$ and $H4$ were identical. However for larger values of H the estimator $H3$ was noticeably more stable. The numerical characteristics of the estimator $H5$ surpassed those of $H1 - H4$ for both considered processes, yet in case of (**I**) there were multiple outliers. The magnitudes of these outliers were directly proportional to the lengths of the sample paths. These observations are further illustrated by figures **??** and **??** presenting, respectively, the comparison of the estimators $H1$, $H3$ and $H5$ (for both (**I**) and (**II**)) for the fixed values of H and for the fixed sample path lengths n^2+1.

		$MSE(Hi) \cdot 10^3$						$\mathbf{E}\lvert Hi - H\rvert \cdot 10$			
n	H	0.55	0.65	0.85	0.95	n	H	0.55	0.65	0.85	0.95
	H1	33.21	36.35	26.61	22.93		H1	1.439	1.529	1.262	1.151
	H2	33.38	30.77	26.10	19.64		H2	1.428	1.403	1.238	1.133
50	H3	9.127	9.303	7.228	6.244	50	H3	0.748	0.778	0.670	0.645
	H4	8.983	10.59	8.874	9.117		H4	0.750	0.809	0.744	0.771
	H5(I)	1.863	0.900	0.379	4.831		H5(I)	0.074	0.068	0.054	0.094
*	H5(II)	0.419	0.390	0.207	0.067		H5(II)	0.017	0.016	0.012	0.007
	H1	13.60	11.34	6.307	7.395		H1	0.926	0.854	0.625	0.696
	H2	12.38	9.380	6.057	4.894		H2	0.874	0.810	0.624	0.554
150	H3	2.506	2.047	1.418	1.043	150	H3	0.395	0.340	0.299	0.259
	H4	2.594	2.216	2.013	1.717		H4	0.401	0.351	0.355	0.322
	H5(I)	0.009	0.024	0.159	0.042		H5(I)	0.007	0.010	0.019	0.011
*	H5(II)	0.030	0.027	0.017	0.006		H5(II)	0.004	0.004	0.003	0.002
	H1	4.128	3.337	2.941	1.970		H1	0.525	0.463	0.435	0.357
	H2	3.176	2.581	1.721	1.573		H2	0.454	0.400	0.334	0.318
500	H3	0.372	0.334	0.232	0.180	500	H3	0.148	0.138	0.118	0.108
	H4	0.382	0.365	0.278	0.290		H4	0.151	0.145	0.129	0.132
	H5(I)	0.000	0.003	0.015	0.029		H5(I)	0.003	0.004	0.008	0.010
*	H5(II)	0.008	0.007	0.005	0.002		H5(II)	0.002	0.002	0.002	0.001

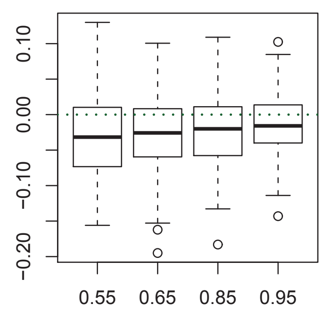
Table 1: Comparison of the estimators



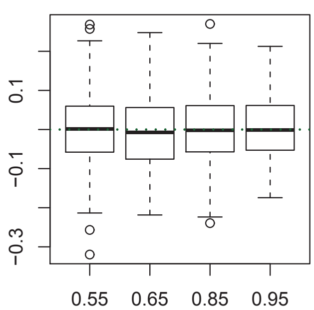
(a) H1, n=50



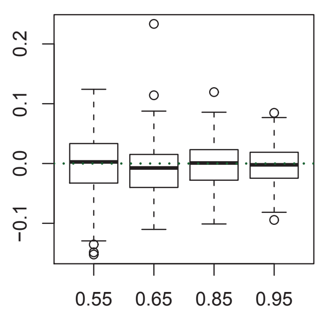
(b) H1, n=150



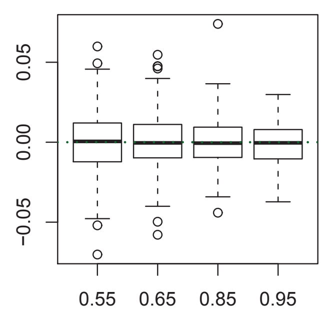
(c) H1, n=500



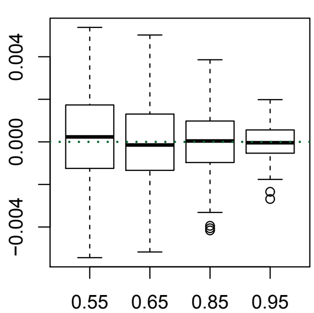
(d) H3, n=50



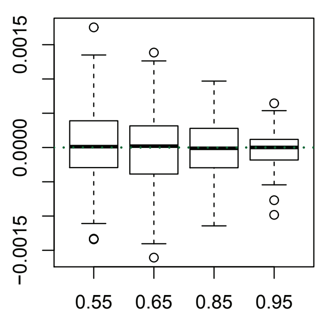
(e) H3, n=150



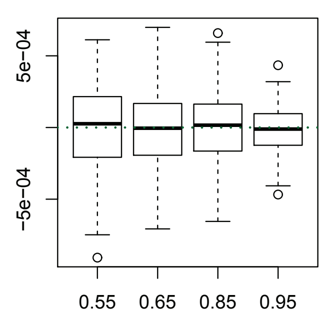
(f) H3, n=500



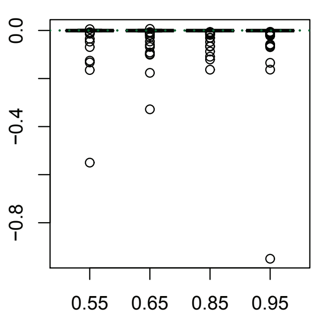
(g) H5(II), n=50



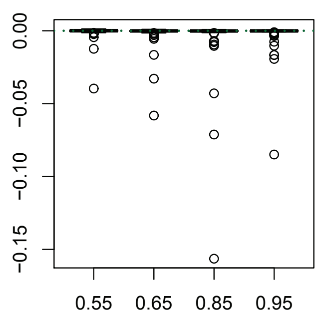
(h) H5(II), n=150



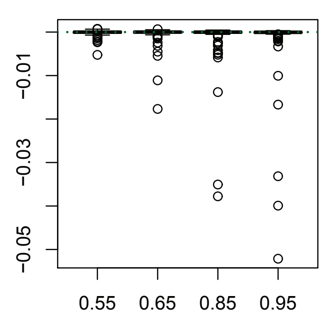
(i) H5(II), n=500



(j) H5(I), n=50

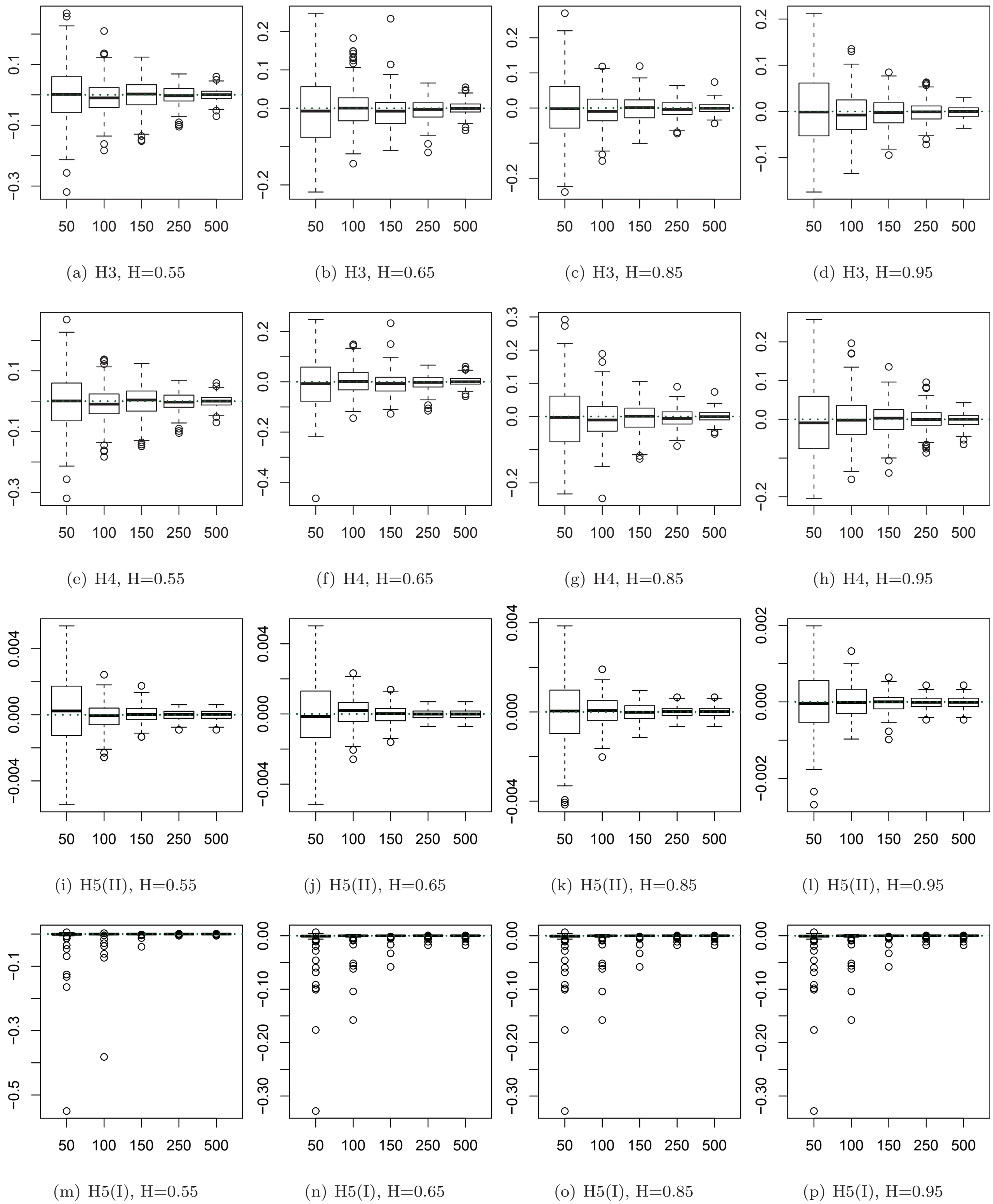


(k) H5(I), n=150



(l) H5(I), n=500

Figure 1: the dependance of $(Hi - H)$ on the value of H



(a) H3, H=0.55 (b) H3, H=0.65 (c) H3, H=0.85 (d) H3, H=0.95

(e) H4, H=0.55 (f) H4, H=0.65 (g) H4, H=0.85 (h) H4, H=0.95

(i) H5(II), H=0.55 (j) H5(II), H=0.65 (k) H5(II), H=0.85 (l) H5(II), H=0.95

(m) H5(I), H=0.55 (n) H5(I), H=0.65 (o) H5(I), H=0.85 (p) H5(I), H=0.95

Figure 2: the dependance of $(Hi - H)$ on the length of the sample paths $n^2 + 1$

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